On the Group Front and Group Velocity in a Dispersive Medium Upon Refraction From a Nondispersive Medium

Conventional definitions of velocities associated with the propagation of modulated waves cannot clearly describe the behavior of the wave packet in a multidimensional dispersive medium. The conventional definition of the phase velocity, which is perpendicular to the wave front, is a special case of the generalized phase velocity defined in this work, since there exist an infinite number of solutions to the equation describing the wave-front movement. Similarly, the generalized group-front velocity is defined for the movement of a wave packet in an arbitrary direction. The group-front velocity is the smallest speed at which the group-front travels in the direction normal to the group front. The group velocity, which is the velocity of energy flow in a nondissipative medium, also satisfies the group-front equation. Because the group-front velocity and the group velocity are not always the same, the direction in which the wave packet travels is not necessarily normal to the group front. In this work, two examples are used to demonstrate this behavior by considering the refraction of a wave packet from vacuum to either a positive-index material (PIM) or a negative-index material (NIM).

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1 Introduction

In reality, all radiation propagates in a group of electromagnetic waves of different frequencies. In a nondispersive medium, the velocity at which the wave group travels is the same as the phase speed of each individual monochromatic wave. In a dispersive medium, however, individual waves travel with different phase speeds, resulting in interference effects that continuously change the amplitude of the wave group. The group amplitude is modulated as a wave with its own front, called group front, which travels at its own frequency and speed. The modulation of the group amplitude forms an envelope or wave packet. The speed at which the wave packet travels is called the group velocity, which is generally considered as the velocity of energy transport or signal propagation, with exceptions in anomalous dispersion media [1]. Therefore, the propagation of the wave group is important for radiant energy transfer in such applications as laser-materials interactions and optical diagnostics.

There has been a long history of distinguishing the group velocity from the phase velocity for waves propagating in one dimension. Hamilton formulated the concept of group velocity as early as 1839 [2], which was further developed by Stokes [3], Rayleigh [4], and Havelock [5]. Brillouin [6] published a monograph on the wave propagation and group velocity. Born and Wolf [1] suggested the three-dimensional definition of the group velocity that is consistent with that in the work of Brillouin. Lighthill [7] and Pedlosky [8] provided the three-dimensional definition of the group velocity, which is different from the definition used by Born and Wolf [1] and Brillouin [6]. Since the wave-group propagation was mostly studied in one-dimensional cases, the confusion had not caused serious problem until the direction of the refraction in a negative-index material was considered.

The publication of the experimental results for a structured metamaterial [9], with a negative refractive index in a narrow microwave region, excited tremendous interests to the so-called negative-index material (NIM). Veselago [10] was the first to theoretically predict the behavior of an NIM, also called left-handed material. He postulated that light would be refracted negatively from a positive-index material (PIM) to an NIM; and furthermore, a plane slab of NIM could focus light. Pendry [11] went further to claim that the NIM could be used to make a perfect lens that would focus an image with a resolution not restricted by the diffraction limit, which is about half of the wavelength. It has also been shown that a layer of NIM could enhance the energy transmittance through evanescent waves [12,13], suggesting that materials with negative refractive indexes may be used to construct microscale energy conversion devices.

Valanju et al. [14] cast doubt on the negative refraction of light. They pointed out that a real beam that has a finite spread of frequencies would be refracted “positively” in an NIM, even though the phase would be refracted “negatively.” Smith et al. [15] argued that Valanju et al. wrongly identified the group velocity as the direction of the interference pattern movement. A thorough understanding of wave-group propagation and refraction is required in order to resolve this dispute. It turns out that much of the confusion arose from inconsistent definitions of the group velocity in multidimensions. It becomes necessary to re-examine the conventional definitions of velocities associated with a single wave and a wave packet, and introduce new concepts for clarity.

The present work deals with the propagation of a wave group, especially when it is incident on a dispersive medium (either PIM or NIM) from a nondispersive medium. Each medium is homogeneous and isotropic, with negligible loss or dissipation. The phase velocity will be discussed first with the introduction of the “generalized phase velocity.” In order to clearly describe the movement of the group front, the “generalized group-front velocity” needs to be used. It will be shown that both the group-front velocity and the group velocity are special cases of the generalized group-front velocity; however, in general they are not the same. With these concepts, the refraction of a modulated wave (or wave group) from vacuum to a dispersive medium, for cases with a PIM and an NIM, is investigated.
2 Theory

2.1 The Generalized Phase Velocity. The phase velocity is a basic concept in wave propagation. It has been conventionally accepted as the speed at which the plane of a constant phase, called wave front or phase front, propagates in the direction of the wavevector \( \mathbf{k} \) [8]. This definition is undoubtedly justified in the one-dimensional case. The one-dimensional time-harmonic wave can be expressed kinematically as

\[
E(t,x) = E_0 \cos(kx - \omega t) \tag{1}
\]

where \( E_0 \) is the amplitude and \( \omega \) the angular frequency of the wave. The phase speed can be obtained by differentiating the wave-front equation, \( kx - \omega t = \text{const.} \), with respect to time. Hence,

\[
v_p = \frac{dx}{dt} = \frac{\omega}{k} \tag{2}
\]

The direction of the phase velocity in the one-dimensional case is apparently the same as that of wave propagation, as illustrated in Fig. 1. When wave propagation is extended to two-dimensional or three-dimensional space, however, the conventional definition does not clearly describe the movement of the wave front. Equation (1) is a simplified form of the plane scalar wave equation in the vector space,

\[
E(t,r) = E_0 \exp[-i(\omega t - \mathbf{k} \cdot \mathbf{r})] \tag{3}
\]

The differentiation of the wave-front equation with respect to time yields

\[
\frac{d\mathbf{r}}{dt} \cdot \mathbf{k} - \omega = 0 \tag{4}
\]

Unless \( d\mathbf{r}/dt \) is perpendicular to \( \mathbf{k} \), the above equation has an infinite number of solutions. As pointed out by Tailleux [16], the movement of the wave front can be interpreted with innumerable velocities, not with just one velocity as in the one-dimensional case. Therefore, the generalized phase velocity is defined in the present paper as follows:

\[
v^{(s)}_p = \frac{\omega}{k} \tag{5}
\]

where the superscript "s" indicates an arbitrary direction specified by the unit vector \( \hat{s} \).

The phase velocity in one-dimensional space can be extended to multidimensional space as the smallest speed at which the wave front travels. Since the shortest distance between two wave crests is the one perpendicular to the wave front, the phase velocity is in the same direction as the wavevector \( \mathbf{k} \). Figure 2 shows a plane wave propagating in two-dimensional space, i.e., the \( x-y \) plane, where wave fronts are illustrated with parallel lines. Substituting \( \hat{s} \) in Eq. (5) by the unit wavevector \( \mathbf{k}/k \) yields

\[
v^{(s)}_p = \frac{\omega}{\mathbf{k} \cdot \hat{s}} \tag{6}
\]

which is the universal definition of the phase velocity [1]. The magnitude of the phase velocity is \( v_p = \omega/k \), which is consistent with the phase speed in the one-dimensional case. From Eq. (6), the \( x \)-component of the phase velocity is

\[
v_{p,x} = \frac{\omega k_x}{k} \tag{7}
\]

It should be noted that \( v_{p,x} \) is not the same as the magnitude of the generalized phase velocity in the \( x \)-direction \( v^{(s)}_p \), which can be found from Eq. (5) as

\[
v^{(s)}_p = \frac{\omega}{k_x} \tag{8}
\]

Clearly, \( v^{(s)}_p = v_p \neq v_{p,x} \). As illustrated in Fig. 2, the wave front moves faster to the observer looking towards the \( x \)-direction than to the one looking towards the \( \hat{s} \)-direction. The generalized phase velocity in the \( x \)-direction has often been misinterpreted as the \( x \)-component of the phase velocity in the literature. As remarked by Pedlosky [8], “the phase speed does not satisfy the rule of vector composition.” The dilemma of vector composition is resolved by distinguishing the generalized phase velocity in the \( x \)-direction from the \( x \)-component of the phase velocity.

2.2 The Generalized Group-Front Velocity. The concept of the group front (or interference front) comes from the interference between waves of different frequencies in a wave group (or wave packet). For the sake of simplicity without loss of generality, two plane waves of the same amplitude with slightly different frequencies are used to demonstrate the movement of the group front. The interference of the two waves forms a wave packet (or a modulated wave), which can be represented by the superposition of the two monochromatic waves,

\[
E(t,r) = E_0 \exp[-i(\omega - \delta \omega - t - \mathbf{k} \cdot \mathbf{r})] + E_0 \exp[-i(\omega + \delta \omega - t - \mathbf{k} \cdot \mathbf{r})] \tag{9}
\]

where \( \omega = \omega - \delta \omega/2, \omega = \omega + \delta \omega/2, \mathbf{k} = \mathbf{k} - \delta \mathbf{k}/2, \) and \( \mathbf{k} = \mathbf{k} + \delta \mathbf{k}/2 \) with a small and positive \( \delta \omega \). Equation (9) can be re-written as follows:

\[
E(t,r) = 2E_0 \cos\left[\frac{1}{2}(\mathbf{k} \cdot \mathbf{r} - t \delta \omega)\right] \exp[-i(\omega t - \mathbf{k} \cdot \mathbf{r})] \tag{10}
\]

The cosine term in Eq. (10) describes the change in the amplitude of the wave group. The constant-amplitude surface, or group front, is defined by

\[
\delta \mathbf{k} \cdot \mathbf{r} - t \delta \omega = \text{const.} \tag{11}
\]

Differentiating Eq. (11) with respect to time gives

\[
\delta \mathbf{k} \cdot \frac{d\mathbf{r}}{dt} - \delta \omega = 0 \tag{12}
\]
The above equation is of the same form as Eq. (4) and has an infinite number of solutions. Hence, it is natural to introduce the generalized group-front velocity,

\[ v_{gf}^{(+)} = \frac{\delta \omega}{\delta k} \hat{s} \]  

(13)

which is defined in an arbitrary direction specified by the unit vector \( \hat{s} \).

Similarly, the group-front velocity represents the orthogonal propagation of the group front and is the smallest speed at which the group front travels. It can be shown from Eq. (13) that the group-front velocity is

\[ v_{gf} = \frac{\delta \omega}{\delta k} \]  

(14)

where \( \delta k = (\delta k_x)^2 + (\delta k_y)^2 + (\delta k_z)^2 \) \( 1/2 \). The group-front velocity is in the same direction as \( \delta k \). If all the waves in the wave group travel in the same direction, then \( \delta k \) and \( k \) are either parallel or antiparallel in an isotropic medium. If the waves travel in different directions or in an anisotropic medium, \( \delta k \) may not be parallel to \( k \). Therefore, the group-front velocity is generally not parallel to \( k \).

The group-front velocity can also be expressed as

\[ v_{gf} = \frac{\delta k}{\delta k} \left[ \frac{\delta k_x}{\delta \omega} + \frac{\delta k_y}{\delta \omega} + \frac{\delta k_z}{\delta \omega} \right]^{1/2} \]  

(15)

In the limit of \( \delta \omega \to 0 \), it can be approximated by

\[ v_{gf} \approx \frac{\delta k}{\delta k} \left[ \frac{\delta k_x}{\delta \omega} + \frac{\delta k_y}{\delta \omega} + \frac{\delta k_z}{\delta \omega} \right] \]  

(16)

Equation (15) or (16) was incorrectly identified by some classical texts as the group velocity. Examples are Born and Wolf [1], Eq. (56) in Chap. 1, and Brillouin [6], Eqs. (26) and (27) in Chap. 4.

The failure to identify the group-front velocity as a special case of the generalized group-front velocity and to distinguish it from the group velocity has frequently caused misinterpretation of the direction of energy propagation. The group velocity \( v_g \) and its relation with \( v_{gf} \) are discussed in the subsequent section.

2.3 The Group Velocity. The appropriate definition of the group velocity in a multidimensional space is [7,8,17]

\[ v_g = V_k \omega = \frac{\delta \omega}{\delta k} + \frac{\delta \omega}{\delta k} + \frac{\delta \omega}{\delta k} = \delta k \cdot \hat{s} \]  

(17)

where \( \delta k \) denotes the gradient of \( \omega \) in the wavevector space. Using the dispersion relation, \( \omega = \omega(k) = \omega(k_x, k_y, k_z) \), the frequency perturbation \( \delta \omega \) can be written as

\[ \delta \omega = \frac{\delta \omega}{\delta k} \delta k_x + \frac{\delta \omega}{\delta k} \delta k_y + \frac{\delta \omega}{\delta k} \delta k_z = \delta k \cdot \hat{s} \]  

(18)

The comparison of Eq. (18) with Eq. (12) suggests that the group velocity is also a solution of the group-front equation and, therefore, is one special case of the generalized group-front velocity, Eq. (13). Nevertheless, the group velocity defined by Eq. (17) is a very important physical quantity because it identifies the direction and speed of energy propagation. Bers [18] proved the equality of the group velocity and energy velocity for propagating waves in a linear, nondissipative, and nonmagnetic medium with normal dispersion. The discussion of the group velocity in an anisotropic medium can also be found in Kong’s work [17]. More recently, Ruppin [19] showed that the group velocity coincides with the energy velocity in the frequency region with a negative refractive index when an NIM is concerned. Notice that, under the condition of anomalous dispersion, the group velocity does not represent the velocity of energy flow and may be greater than the speed of light or negative [20]. Loudon [21] derived the energy velocity by dividing the Poynting vector by the total energy density in a dielectric medium and showed that the group velocity is not equal to the energy velocity in the presence of absorption.

The present work is confined in the region with normal dispersion, nondissipative, and isotropic medium. Under these conditions, the group velocity can be expressed as [15]

\[ v_g = \frac{\partial \omega}{\partial k} \]  

(19)

By inserting \( v_{gf} \) into Eq. (12) and then combining it with Eq. (18) to eliminate \( \delta \omega \), an explicit relation between the group-front velocity \( v_{gf} \) and the group velocity \( v_g \) can be written as

\[ \delta k \cdot (v_{gf} - v_g) = 0 \]  

(20)

There are two cases that satisfy the above equation. The first case is when \( v_{gf} \) is the same as the group velocity \( v_g \). This can only happen when \( \delta k \) is either parallel or antiparallel to \( k \), for instance, when the wave group travels in a nondispersive medium or when the beam is incident normally to a dispersive medium from a nondispersive medium. The second case is when the difference vector between \( v_{gf} \) and \( v_g \) is perpendicular to \( \delta k \) (or \( v_g \)).

The group-front velocity \( v_{gf} \) was called the velocity of the interference front by Smith et al. [15], who pointed out that the group velocity can be decomposed into \( v_{gf} \) and another component normal to the group-front velocity.

3 Application to Refraction Phenomenon

This section deals with the refraction of a wave packet by considering two plane waves of same amplitude with slightly different frequencies incident from a nondispersive medium to a dispersive medium. As mentioned before, each medium is homogeneous and isotropic with negligible loss or dissipation. The real value of \( E(t, r) \) in Eq. (10) can be used as the amplitude of the electric field or magnetic field. The wavevector for each medium is

\[ k_j = [k]_j \cdot \sigma = \omega/c_0 n_j, \quad j = 1, 2 \]  

(21a)

where \( c_0 = 2.998 \times 10^8 \) m/s is the speed of light in vacuum, \( n_j \) is the refractive index of the 1 st (\( j = 1 \)) and 2 nd (\( j = 2 \)) medium, and \( \sigma \) is chosen to be +1 for PIMs and -1 for NIMs because \( k_j \) should be positive regardless of the medium [14]. From the phase-matching condition, the \( x \)-component of the wavevector is given by

\[ k_{1,x} = k_{2,x} = \frac{\omega}{c_0}, \quad \text{(21c)} \]

The \( z \)-component of the wavevector in the second medium is

\[ k_{2,z} = \sigma \sqrt{k_{2,x}^2 - k_{1,x}^2} = \sigma \frac{\omega}{c_0} \sqrt{n_2^2(\omega) - n_1^2} \sin \theta_1 \]  

(21d)

Since the second medium is dispersive, \( n_2 \) is a function of frequency, whereas \( n_1 \) is taken as a constant to represent the nondispersive medium. Note that the present work deals with propagating waves only, that is, \( \sqrt{k_{2,z}^2 - k_{2,x}^2} \) in Eq. (21d) is always real and positive.

The phase velocity in a dispersive medium can be calculated from Eqs. (6) and (21b), yielding

\[ v_{p,2} = \frac{c_0}{n_2} \]  

(22)

and

\[ \tan \theta_p = \frac{k_{2,x}}{k_{2,z}} \frac{n_1 \sin \theta_1}{\sigma \sqrt{n_2^2 - n_1^2}} \]  

(23)
where \( \theta_p \) is the angle between \( \mathbf{k} \) and the \( z \)-axis. Equation (23) is sometimes called the generalized Snell’s law [22]. Figure 3(a) represents the geometry when the second medium is a PIM. Since the increase of \( \omega \) in the normal dispersion region (\( dn/d\omega > 0 \)) causes a decrease of \( v_{p,2} \) and \( \theta_p \), the wave with a slightly higher frequency (\( \omega^+ \)) propagates closer to the \( z \)-axis than the other in the PIM. In contrast, as shown in Fig. 3(b), an NIM case gives a different tendency. According to Eq. (23), \( \theta_p \) is between \( \pi/2 \) and \( \pi \) for a negative refractive index. That is, wave fronts in the NIM travel toward the interface and merge with those propagating in the first medium. When \( \omega \) increases in the normal dispersion region, \( v_{p,2} \) increases and \( \theta_p \) and \( k_2 \) decrease. Hence, the \( \omega^- \) wave is closer to the \( z \)-axis than the \( \omega^+ \) wave.

From Eqs. (19) and (21b), the group velocity in the second medium becomes:

\[
v_{g,2} = \frac{c_0}{n_2 + \omega \frac{dn_2}{d\omega}}
\]

(24)

where

\[
v_{g,2} = \frac{c_0}{n_2 + \omega \frac{dn_2}{d\omega}}
\]

(25)

The direction of the group velocity upon refraction is determined by

\[
\tan \theta_g = \frac{k_{2,2}}{k_{2,2}} = \tan \theta_p
\]

(26)

Therefore, \( \theta_g \) must be equal to \( \theta_p \) or \( \theta_p - \pi \). The direction of the group velocity depends on the type of medium in the second medium. For a PIM, \( \theta_p \) is identical with \( \theta_p \), because the Poynting vector is parallel to the wavevector \( \mathbf{k} \) of a plane wave. For an NIM, however, the direction of the group velocity is anti-parallel to the wavevector according to \( \sigma \) in Eq. (24), which indicates that \( \theta_g \) should be negative (or \( \theta_g = \theta_p - \pi \)). Hence, it is apparent that the velocity of energy flow, as determined by the group velocity, is refracted negatively into the NIM and propagates in the opposite direction of the wavevector. It is consistent with Veselago’s claim that the Poynting vector propagates in the opposite direction of the wavevector in NIMs [10]. In addition, one can define the group index according to Eq. (25) as

\[
n_g = n_2 + \omega \frac{dn_2}{d\omega} = \frac{d(\omega n_2)}{d\omega}
\]

(27)

which should be greater than unity regardless of the sign of the refractive index of the medium, as required by the principle of causality (i.e., an effect cannot occur before the cause). Note again that, in the anomalous dispersion region, the group index may be less than unity or even negative, and the group velocity no longer represents the propagation of energy.

In order to obtain the group-front velocity, it is necessary to derive the equation for \( \delta k_2 \), first. Since each component of the wavevector \( \mathbf{k}_0 \) is a function of \( \omega \), it can be approximated to the following sets of equations for small \( d\omega \):

\[
\delta k_2 = \delta k_{2,x} + \delta k_{2,z}
\]

(28)

where

\[
\delta k_{2,x} = \frac{\delta \omega}{c_0} n_1 \sin \theta_1
\]

(29)

and

\[
\delta k_{2,z} = \frac{\delta \omega}{c_0} \left[ \sqrt{n_2^2 - n_1^2 \sin^2 \theta_1} + \frac{n_2}{\sqrt{n_2^2 - n_1^2 \sin^2 \theta_1}} \right]
\]

(30a)

Rearranging the \( z \)-component of \( \delta k_2 \) yields

\[
\delta k_{2,z} = \frac{\delta \omega}{c_0} \frac{n_2 n_1 \sin \theta_1}{\sqrt{n_2^2 - n_1^2 \sin^2 \theta_1}}
\]

(30b)

which is always positive in the normal dispersion region regardless of the type of the medium. The direction of the group-front velocity in the dispersive medium is derived from Eqs. (29) and (30b) as

\[
\tan \theta_{gf} = \frac{\delta k_{2,z}}{\delta k_{2,z}} = \frac{n_1 \sin \theta_1}{\frac{n_2 n_1 \sin \theta_1}{\sqrt{n_2^2 - n_1^2 \sin^2 \theta_1}}}
\]

(31)

This equation was named the group Snell’s law by Valanju et al. [14]. Equation (31) indicates that the group front is always oriented to the positive direction, even in a negative-index medium.

When the second medium is nondispersive, the phase velocity \( v_p \), group-front velocity \( v_{gf} \), and group velocity \( v_g \) are the same. This can happen only in a PIM because causality requires that all NIMs be dispersive [10]. The group front is parallel to the individual wave front. When the second medium is dispersive and radiation is incident normally (\( \theta_1 = 0 \)) to the interface, \( v_p \) is the same as \( v_{gf} \) and both are in the positive \( z \)-direction (forward propagating). Their magnitude is \( v_p = c_0 / n_1 \). Both the group front and the phase front are parallel in the \( x-y \) plane. The phase velocity may be in the positive \( z \)-direction (PIM) or in the negative \( z \)-direction (NIM) with a magnitude \( v_p = \sigma c_0 / n_2 \) that is different from \( v_g \). In general, the phase velocity, the group-front velocity, and the group velocity possess different directions and magnitudes upon refraction into a dispersive medium at oblique incidence. It is important to distinguish these velocities in order to fully understand wave-group propagation.
4 Results and Discussion

The behavior of wave-packet refraction into a dispersive medium is illustrated in this section for two cases, one with a PIM and the other with an NIM as the second medium. Even though only one specific incidence angle from vacuum onto the second medium is considered, these two cases represent the general features of wave-packet refraction because variations in the incidence angle and/or materials merely affect the quantities, such as the refracted direction of each velocity and spacing between adjacent fronts, without changing the general trend.

The locus of a wave packet, consisting of two monochromatic waves described by Eq. (9), is shown in Fig. 4 when the wave is incident onto a dispersive PIM from vacuum \((n_1=1.0)\) at an incidence angle \(\theta_i=45\) deg. The PIM is chosen as magnesium oxide (MgO) because of the relatively large frequency-dependence of its refractive index \(dn/d\omega\) and low loss (i.e., little absorption) in the mid-infrared region. The refractive index \(n_3=1.395\) at the wavelength \(\lambda_0=9.091\ \mu m\) \((\omega=2.072\times10^{14}\ \text{rad/s})\) is taken from Ref. [23]. At this wavelength, the extinction coefficient is less than 0.005, which is negligible when dealing with wave propagation near the surface. In the calculation, the interval between the frequencies of the two monochromatic waves is chosen to satisfy \(\delta\omega/\omega = 0.1\). The group index calculated from Eq. (27) is \(n_g = 2.488\). A uniform time step \(\Delta t\) is used for drawing wave fronts and group fronts.

In the first medium wave fronts and group fronts are the same and the two waves propagate in the same direction. When the waves are refracted into the second medium, they propagate into different directions. The angle between two refracted waves is \(\Delta \theta_p = 2.65\) deg; therefore, the distinction of the two wavevectors \(\mathbf{k}_2 = \mathbf{k}_1 - \partial \mathbf{k}/\partial \omega\) and \(\mathbf{k}'_2 = \mathbf{k}'_1 + \partial \mathbf{k}/\partial \omega\) is not very sharp. However, the group front and the wave front are apparently not parallel to each other. The group velocity \(\mathbf{v}_g\) is parallel to \(\mathbf{k}_2\), while the group-front velocity \(\mathbf{v}_{gf}\) is parallel to \(\partial \mathbf{k}/\partial \omega\). The angle between \(\mathbf{v}_g\) and the \(z\)-axis is \(\theta_g = 30.44\) deg, and that between \(\mathbf{v}_{gf}\) and the \(z\)-axis is \(\theta_{gf}=15.97\) deg.

Because group fronts are not parallel to phase fronts, an inhomogeneous wave exists in the second medium [1]. It is very interesting to see how the individual waves and the wave group propagate into the dispersive medium. Since wave fronts for the two individual waves are hardly distinguishable, in order to avoid confusion, Fig. 4 only shows averaged phase fronts that propagate in the direction of the mean wavevector \(\mathbf{k}_1\). Group fronts, on the other hand, are perpendicular to the direction of the differential wavevector \(\partial \mathbf{k}/\partial \omega\). The spacing between adjacent wave fronts is greater than that between adjacent group fronts, indicating that the group front moves forward with a slower speed.

Consider wave front AB that is refracted into the second medium. When point A reaches point C, point B will reach point \(D_p\), as shown in Fig. 4. The small squares at points A and \(D_p\) indicate the orthogonal relationship between the wavevector and the wave front. The square at a point \(D_p\) indicates the orthogonal relationship between \(\partial \mathbf{k}_2\) and the group front. If AB is considered as the group front, the group front will propagate from AB to \(D_p\) rather than CD. That is, the group front moves in a skewed direction according to the group velocity (from AB to \(D_p\)), not according to the group-front velocity (from AB to CD), which is normal to the group front. Notice that although the group index in Eq. (27) directly affects the speed of wave-group propagation (i.e., the magnitude of the group velocity), the group index has no relations with the direction of wave-group propagation (i.e., the direction of the group velocity). Instead, one must use the phase-matching condition to determine the angle of refraction or the direction of the group velocity. Notice that Eq. (31) should be used to determine the orientation of the group front. Although Smith et al. [15] was the first to point out that the group front moves in the direction of the group velocity that is not perpendicular to the group front in an NIM, it is difficult to experimentally confirm their theory because of the current limitations in fabricating NIMs with desired properties (e.g., isotropic and lossless). The analysis here using a dispersive PIM demonstrates the same phenomenon that can be experimentally validated with existing materials and instrumentation.

The behavior of wave-packet refraction into an NIM is shown in Fig. 5. For the refractive index of an NIM and its frequency dependence in the microwave region, the permittivity and permeability formulas and parameters are adopted from [24]. The calculated refractive index is \(n_3 = -1.262\) at \(\lambda_0=5.996\ \mu m\) \((\omega = 3.14 \times 10^{10}\ \text{rad/s})\). The absorption in the second medium is neglected. The angle of incidence is fixed at \(\theta_i=45\) deg. For \(\delta\omega/\omega = 0.01\), the angle between two refracted waves is about \(\Delta \theta_p = 2.37\) deg. The time step \(\Delta t\) for this figure is slightly larger than that used in Fig. 4. Again, only mean wave fronts are shown. Figure 5 clearly shows that group fronts are not parallel to the phase fronts in the NIM. The calculated group index is \(n_g = 6.467\); this means that the wave group will travel with a speed about 15.5 percent of the speed of light in free space. The spacing between adjacent group fronts is much closer in the NIM.

The wave front moves upwards in the NIM. For example, during the same time when the wave front in vacuum travels from AP to AO, the average wave front in the NIM moves from \(D_pQ\) to \(D_pO\). Notice that the group velocity \(\mathbf{v}_{g,2}\) is antiparallel to \(\mathbf{k}_2\). The \(x\)-component of \(\mathbf{v}_{g,2}\) is negative and \(\theta_{gf}=-34.70\) deg, suggesting that the wave packet is indeed refracted negatively into the NIM.
Group front AB in the first medium will travel to CDg rather than CDg in the second medium. However, the speed for energy propagation is always positive. Thus, negative refraction does not violate the causality principle. Our result agrees with that of Smith et al. [15], who claimed that the wave packet will propagate not only in the direction parallel to $v_{gf}$ but also in the direction normal to $v_{gf}$. They also argued in the same paper that Valanju et al. [14] had misinterpreted the group-front velocity $v_{gf}$ as the velocity of energy flow. Although Valanju et al. later admitted this oversight in their reply to Pendry and Smith’s comments, they insisted that $v_{gf}$ should be the group velocity [25]. Note that the claim of Valanju et al. [14] that the wave in NIM is inhomogeneous still holds. The existence of inhomogeneous waves in the NIM could have an impact on its ability to focus light. Loss or absorption is another practical issue to be addressed and it is expected that high-quality NIMs will be fabricated as the technology advances and NIM can be realized in the infrared and shorter wavelength regions.

5 Conclusions

For a wave propagating in multidimensional space, there exist an infinite number of solutions to the wave-front equation. The present work introduces the concept of the generalized phase velocity, with which the phase velocity is extended to multidimensional space without losing consistency with existing publications. The phase velocity is the smallest speed at which the wave front travels and is always parallel to the wavevector. Furthermore, it obeys the rule of vector composition; however, its component is in general different from the generalized phase velocity in that direction.

The solution of the group-front equation in multidimensions is also not unique. The generalized group-front velocity is defined here and can be in any arbitrary direction; whereas the group-front velocity and the group velocity are two special cases, which are not identical in general. Nevertheless, these two velocities are often confused or used incorrectly in the literature. It is the group velocity that describes the direction and speed of energy flow in a lossless medium. This clarification may help understand the energy transport mechanisms in microsystems.

The difference between the group-front velocity and the group velocity in a dispersive PIM, upon refraction from a nondispersive medium, is demonstrated for the first time. This allows experimental validation to be performed in the near future, because many dielectric materials exhibit normal dispersion in a large spectral region. The wave packet propagates with the group velocity, which can be decomposed into a component normal to the group front (identified as the group-front velocity) and another component parallel to the group front; this is true not only in an NIM but also in a normally dispersive PIM.

It may be useful to extend the present work, which is valid for isotropic and lossless media, to anisotropic and/or dissipative media. Another issue to be addressed in the future is wave propagation in the anomalous dispersion region, where absorption is strong. Experimental studies are also needed to better understand the behavior of inhomogeneous waves, especially in NIMs, which are significantly dispersive.

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Nomenclature

- $c_0$ = speed of light in vacuum
- $E$ = wave field
- $k$ = wavevector
- $n$ = refractive index
- $r$ = space vector
- $\mathbf{s} = \text{unit vector in arbitrary direction}$
- $t = \text{time}$
- $\Delta t = \text{time step}$
- $v = \text{velocity}$
- $v_{p}(x) = \text{generalized phase velocity}$
- $v_{gf} = \text{generalized group-front velocity}$

Greek Symbols

- $\gamma = \text{scattering rate}$
- $\delta = \text{small difference in value}$
- $\varepsilon = \text{relative electric permittivity}$
- $\theta = \text{angle}$
- $\lambda_0 = \text{wavelength in vacuum}$
- $\mu = \text{relative magnetic permeability}$
- $\sigma = \text{sign which represents the type of the medium}$
- $\omega = \text{angular frequency}$

Subscripts

- $1, 2 = \text{first medium and second medium}$
- $g = \text{group}$
- $gf = \text{group front}$
- $p = \text{phase}$
- $x, y, z = \text{components}$

References